Finite Element Method (FEM)

The finite element method (FEM) is the oldest numerical technique applied to engineering problems. FEM itself is not rigorous, but when combined with integral equation techniques it can yield rigorous formulations. Advantages of FEM:

1. Sparse matrices result (as opposed to MM for which dense matrices result). Sparse matrices allow the application of a wide range of fast matrix solvers.

2. Its application involves discretization of the computational domain, and therefore is adaptable to a wide range of geometries and material variations.

Traditional applications like civil and mechanical engineering use scalar node basis functions. Vector edge basis functions are more appropriate for electromagnetic problems because

- EM problems require solutions of vector quantities
- Volume currents are required, not just surface currents
- Continuity and boundary conditions are applied to edges, not just nodes (points)
- Spurious solutions occur with nodes

FEM Formulation (1)

The figure illustrates the generic problem. The computational region is the enclosed volume, Ω. The vector wave equation is the starting point for the FEM solution

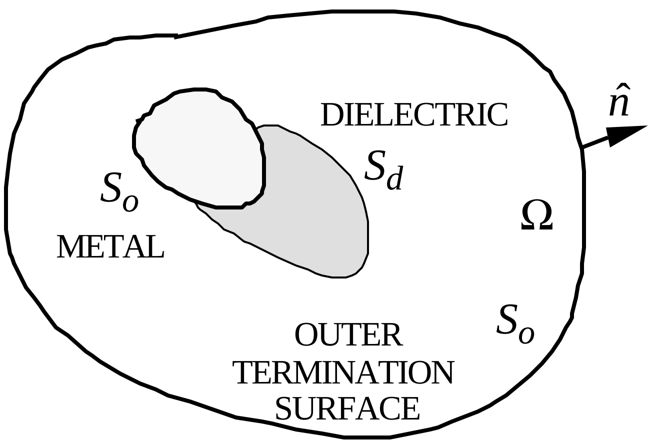
$$\underbrace{\nabla \times \left(\frac{\nabla \times \vec{E}}{m_r} \right) - k_o^2 e_r \vec{E}}_{\equiv L(\vec{E})} = \underbrace{- jk_o Z_o \vec{J} - \nabla \times \left(\frac{\vec{J}_m}{m_r} \right)}_{\equiv \vec{f}}$$

A testing procedure is used similar to the method of moments. Each side is multiplied by a testing (weighting) function, and integrated. Using the inner product notation

$$\left\langle \vec{A}, \vec{W} \right\rangle = \int_{\Omega} \vec{A} \bullet \vec{W} \, d\Omega$$

where $\vec{W}$ is the test function and $\vec{A}$ a field or current. Ideally, the two sides of the wave equation should be equal and the difference zero. In practice the difference will not be zero, so we minimize the functional

$$F(\vec{E}) = \left\langle L(\vec{E}), \vec{W} \right\rangle - \left\langle \vec{f}, \vec{W} \right\rangle$$



FEM Formulation (2)

Green's first vector identity is used to eliminate the double curl. The result is referred to as the <u>weak form</u> of the wave equation. Testing the weak form gives the following equation:

$$\int_{\Omega}\left[\frac{1}{m_r}(\nabla\times\vec{E})\bullet(\nabla\times\vec{W})-k_o^2 e_r\vec{E}\bullet\vec{W}\right]d\Omega-\oint_{S}\frac{1}{m_r}(\hat{n}\times\nabla\times\vec{E})\bullet\vec{W}\,ds+\int_{\Omega}\vec{f}_i\bullet\vec{W}\,d\Omega=0$$

where $\vec{f}_i = 0$ for scattering problems, but not for antenna problems. The unknown quantity is the electric field, $\vec{E}$. A dual equation can be derived for $\vec{H}$.

$\vec{E}$ and $\vec{H}$ are the total fields ($\vec{H}=\vec{H}_i+\vec{H}_s$ and $\vec{E}=\vec{E}_i+\vec{E}_s$). The incident fields are known; the scattered fields are unknown.

The numerical solution for the integral equation begins by discretizing the volume into subdomains. The fields will be computed on the boundaries of the subdomains.

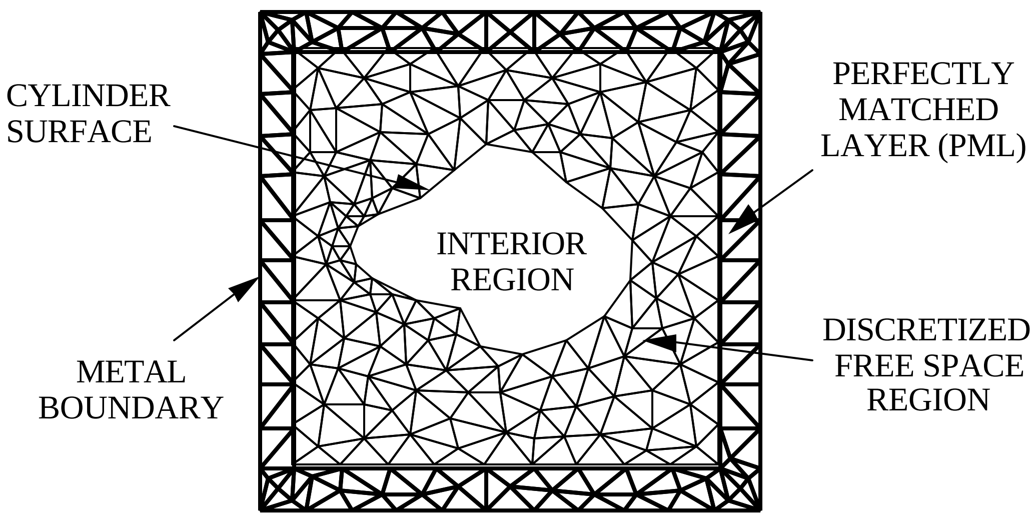
In the equation for $\vec{E}$, integrals over the PEC portions of the surfaces vanish. If the exterior surface (the terminating boundary) is not PEC then a boundary condition must be imposed. (For scattering problems, there should be no reflection at this boundary because it is merely a computational surface, not a physical surface.)

FEM Formulation (3)

The surface integral for the terminating boundary can be handled in several ways:

1. For radiation problems where the source is inside, a perfectly matched layer (PML) can be used just inside of the boundary.
2. The surface integral can be replaced by one that incorporates a general boundary condition.
3. An alternative is to use the method of moments to find the equivalent currents on the outer boundary. However, since the surface currents couple with the fields inside, the MM matrix equation must be solved with the FEM matrix equation.

Example: An infinitely long conducting cylinder (arbitrary cross section) using a PML termination



FEM Formulation (4)

For the discretized volume with a total of N subdomains, each with N_e edges, the scattered field can be expanded into a series of basis functions with unknown expansion coefficients, E_m^e. The field inside subdomain e can be expressed as

$$\vec{E}^e = \sum_{m=1}^{N_e} E_m^e \vec{W}_m^e \quad (e = 1, \ldots, N)$$

where the expansion coefficients are determined by solving the matrix equation:

$$\left[A_{mn}^e\right] \left[E_m^e\right] = \left[B_m^e\right] \text{ or } \bar{\bar{A}}\bar{E} = \bar{B}$$

The overbar is a column vector and the double overbar a two square matrix. The vector and matrix elements are of the form

$$A_{mn}^e = \int_\Omega \left[\frac{1}{m_r} (\nabla \times \vec{W}_m^e) \bullet (\nabla \times \vec{W}_n^e) - k_o^2 e_r \vec{W}_m^e \bullet \vec{W}_n^e \right] d\Omega$$

$$B_m^e = \oint_{S_d} \frac{1}{m_r} (\hat{n} \times \nabla \times \vec{E}_i) \bullet \vec{W}_m^e ds - \int_\Omega \left[\nabla \times \left(\frac{1}{m_r} \nabla \times \vec{E}_i \right) - k_o^2 e_r \vec{E}_i \right] \bullet \vec{W}_m^e \, d\Omega$$

Perfectly Matched Layers (1)

The electrical characteristics of a material are completely described by complex permittivity and permeability matrices

$$\overline{\overline{e}} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} \quad \text{and} \quad \overline{\overline{m}} = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} \\ m_{yx} & m_{yy} & m_{yz} \\ m_{zx} & m_{zy} & m_{zz} \end{bmatrix}$$

Using this notation, Maxwell's equations can be written in matrix form. For example,

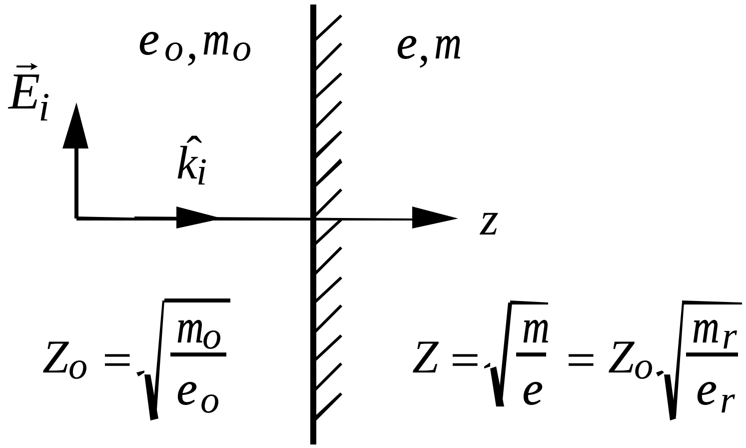
$$\vec{D} = \overline{\overline{e}}\,\vec{E} \quad \Rightarrow \quad D_x = e_{xx}E_x + e_{xy}E_y + e_{xz}E_z, \quad \text{etc. and } \nabla \times \vec{H} = jw\,\overline{\overline{e}}\,\vec{E}$$

Most materials have diagonal permittivity and permeability matrices

$$\frac{\overline{\overline{e}}}{e_o} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

For a homogeneous isotropic medium: $a = b = c$.

Perfectly Matched Layers (2)



Let the permittivity equal the permeability at every point in the material

$$\frac{\overline{\overline{e}}}{e_o} = \frac{\overline{\overline{m}}}{m_o} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

As in the standard solution of the plane wave reflection coefficients, the fields are written for the two media and tangential components equated at the boundary. However, the dispersion relationship must be used to determine under what conditions the reflection coefficients will be zero. The result is[1]

$$R_\perp = R_{\mathrm{TE}} = \frac{\cos q_i - \sqrt{b/a}\cos q_t}{\cos q_i + \sqrt{b/a}\cos q_t}$$

$$R_\parallel = R_{\mathrm{TM}} = \frac{\sqrt{b/a}\cos q_t - \cos q_i}{\cos q_i + \sqrt{b/a}\cos q_t}$$

[1]See S. D. Gedney, "An Anisotropic Perfectly Matched Layer-Absorbing Medium for the Truncation of FDTD Lattices," *IEEE Transactions on Antennas & Propagation*, vol. 44, no. 12, Dec. 1996.

Perfectly Matched Layers (3)

The reflection coefficients will be zero if

$$a = b \quad \Rightarrow \quad R_{\mathrm{TM}} = R_{\mathrm{TE}} = 0$$

and independent of angle if $\sqrt{bc} = 1$. Therefore,

$$\sqrt{bc}\cos\theta_t = \cos\theta_i \quad \Rightarrow \quad \sqrt{bc} = 1$$

Choose $a = b = A - jB$ and $c = \dfrac{1}{A - jB}$ (A and B are real). A good choice is $A = B \approx 1$.

Features of the PML:

1. It is a reflectionless medium for all frequencies and incidence angles
2. Not physically realizable, but it is used as a termination for computational domains in numerical solutions
3. B controls the absorptivity of the layer
4. The PML represents a perfectly matched uniaxial anisotropic medium

Grid Termination Methods (1)

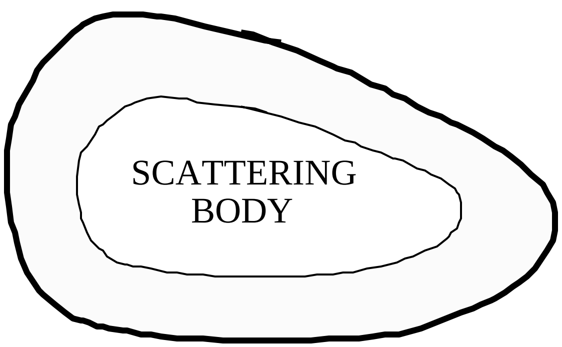
The two most common methods for terminating the computational domain are:

(1) Perfectly Matched Layers (PMLs) – The grid is bounded by a layer of nonreflecting material.

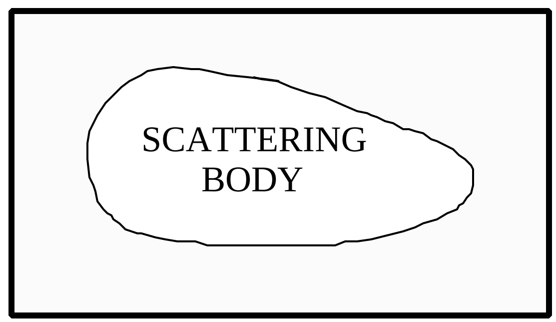
(2) Absorbing Boundary Conditions (ABCs) – A boundary condition is applied to the field at the edge of the grid so that the radiation conditions are satisfied (the transmitted field decays to zero at infinity and no reflection at the computational boundary). ABCs take on various forms and are also referred to as transparent boundary conditions and radiation boundary conditions.

Both can be applied conformally or non-conformally.

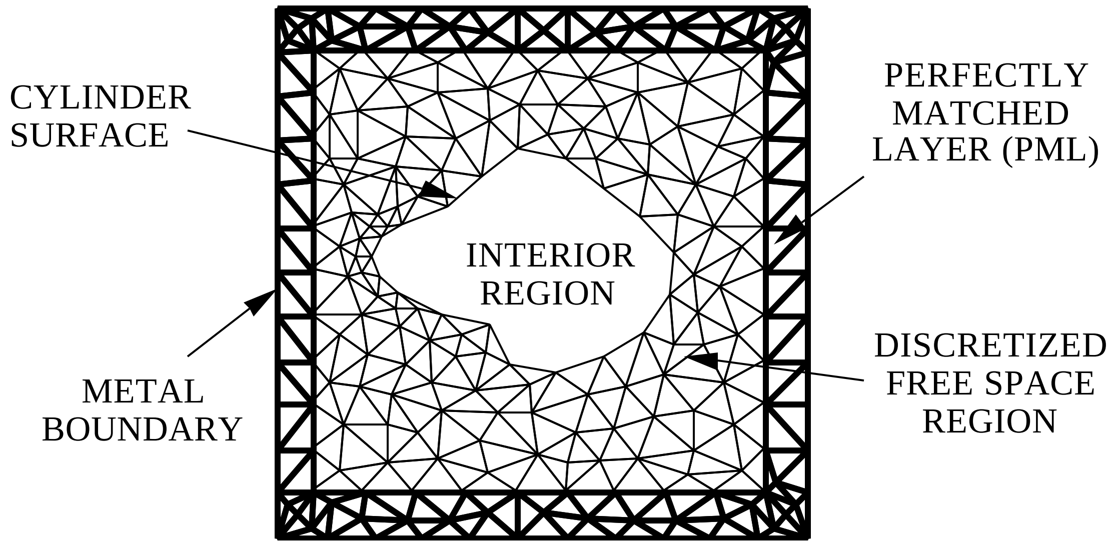
CONFORMAL TERMINATION



NONCONFORMAL TERMINATION



Grid Termination Methods (2)



The grid can be terminated by a perfectly matched layer with a metal backing. The thickness and loss of the layer are determined so that the reflections from it are negligible.

- This approach is convenient but requires additional nodes in the computational domain (i.e., the nodes inside of the PML)
- Typical values are $e_r = m_r = 1 - j$ and PML thickness $d = 0.15\, l_o$
- The closest points on the target should be $1\, l_o - 2\, l_o$ from the PML
- Conformal layers are preferred because they minimized the number of additional nodes

Absorbing Boundary Conditions (1)

The electric and magnetic fields satisfy the vector wave equations

$$\nabla \times \nabla \times \vec{y} - k^2 \vec{y} = 0$$

where $\vec{y}$ is either $\vec{E}$ or $\vec{H}$. All physically realizable fields must decay to zero at infinity. This is the <u>radiation condition</u>, which can be expressed as

$$\lim_{r \to \infty} r\left\{\hat{r} \times \left(\nabla \times \vec{y}\right) - jk\vec{y}\right\} = 0$$

The fields can be expressed as a series of terms in powers of 1/r (the Wilcox expansion theorem)

$$\vec{y}(r, q, f) = \frac{e^{-jkr}}{4p\, r} \sum_{n=0}^{\infty} \left(\frac{\vec{y}_n(q, f)}{r^n} \right)$$

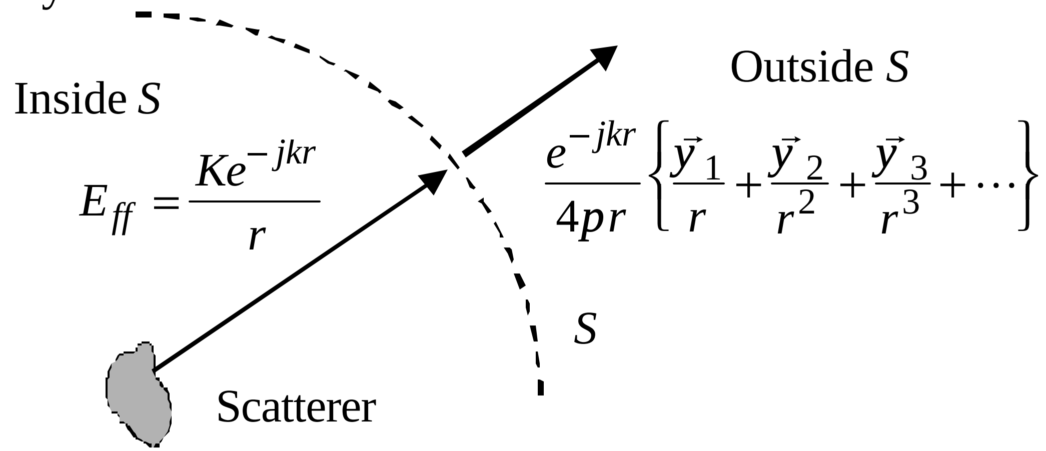
$$= \frac{e^{-jkr}}{4p\, r} \left\{ \vec{y}_0 + \frac{\vec{y}_1}{r} + \frac{\vec{y}_2}{r^2} + \cdots \right\}$$

Using the series in the radiation conditions

$$\hat{r} \times \left(\nabla \times \vec{y}\right) - jk\vec{y} = \frac{e^{-jkr}}{4p\, r} \sum_{n=0}^{\infty} \left(-jk \frac{\hat{r}(\hat{r} \bullet \vec{y})}{r^n} + \frac{r\nabla(\hat{r} \bullet \vec{y}) - n\hat{r} \times (\hat{r} \times \vec{y})}{r^{n+1}} \right)$$

Absorbing Boundary Conditions (2)

If the radiation condition is applied at a finite distance r that is in the far field of the scatterer, then there is no error. The higher order terms will be forced to zero by equating fields on the boundary



If the boundary condition is applied at a distance r that is in the near field, then the boundary condition is in error. Waves will be set up inside S to match the error terms at the boundary. In the above example, the error will be on the order of $1/r^2$ denoted $O(r^{-2})$. Higher order boundary conditions can be derived. For example[2],

$$\left\{\prod_{m=2}^{N}\left(\hat{r}\times(\nabla\times) - jk - 2(n-1)/r\right)\times\left(\hat{r}\times(\nabla\times\vec{y}) - jk\vec{y}_{\tan}\right)\right\} = 0$$

where $\vec{y}_{\tan} = -r\times(r\times\vec{y})$, satisfies the Nth order condition: $O(r^{-2N-1})$.

[2] A. Pederson, "Absorbing Boundary Conditions for the Vector Wave Equation," *Microwave & Optical Tech. Lett.*, vol. 1, pp.62-64.

Absorbing Boundary Conditions (3)

Example: For a second order condition the error is $O(r^{-5})$. The exact field at distance r has an infinite number of terms:

$$\vec{y}_{\text{exact}} = \frac{e^{-jkr}}{4p\, r}\left(\vec{y}_0 + \frac{\vec{y}_1}{r} + \frac{\vec{y}_2}{r^2} + \cdots \right)$$

The boundary condition allows us to retain the terms up to $O(r^{-4})$

$$\vec{y}_{\text{approx}} = \frac{e^{-jkr}}{4p\, r}\left(\vec{y}_0 + \frac{\vec{y}_1}{r} + \frac{\vec{y}_2}{r^2} + \frac{\vec{y}_3}{r^3} \right) + \underbrace{\frac{e^{-jkr}}{4p\, r}\left(\frac{\vec{A}_4}{r^4} + \frac{\vec{A}_5}{r^5} + \cdots \right)}_{\text{Fictitious outgoing waves}} + \underbrace{\frac{e^{jkr}}{4p\, r}\left(\frac{\vec{B}_4}{r^4} + \frac{\vec{B}_5}{r^5} + \cdots \right)}_{\text{Fictitious incoming waves}}$$

Note:

- Low order boundary conditions must be used at large r or significant errors occur. However, this requires a larger computational domain (i.e., more basis functions).
- Higher order boundary conditions can be applied closer to the surface, but by are more complex and difficult to implement.
- Note that an increasing number of derivatives are required as the order increases.

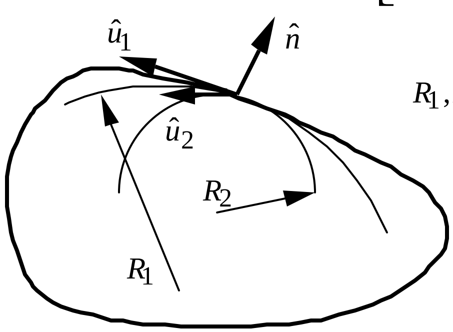
Absorbing Boundary Conditions (4)

There are many forms of absorbing boundary conditions, but in general

1. They are all derived from the radiation conditions as $R \to \infty$
2. ABCs are equivalent to a surface impedance boundary condition
3. The particular form depends on the distance from the scattering surface at which they are applied. Relatively simple equations (first order) can be used when applied far from the target ($1 l_o - 2 l_o$, where l_o is the free space wavelength).
4. ABCs applied on conformal surfaces are more complex than those applied on coordinate variable constant planes

Another example is the first order conformal ABC based on geometrical optics

$$\hat{n} \times \nabla \times \vec{E} - \left[jk_o + \frac{1/R_1 + 1/R_2}{2} - \underbrace{\left(\frac{\hat{u}_1 \hat{u}_2}{R_1} + \frac{\hat{u}_2 \hat{u}_2}{R_2} \right)}_{\text{DYADIC}} \bullet \right] \vec{E}_t = 0$$



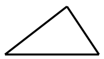
R_1, R_2 =PRINCIPAL RADII OF CURVATURE

R_1, R_2 are surface radii of curvature

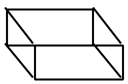
FEM Subdomains (1)

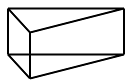
TWO DIMENSIONAL

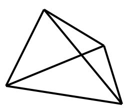
 RECTANGLES

 TRIANGLES

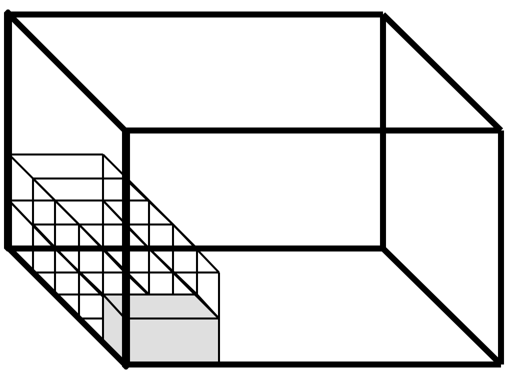
THREE DIMENSIONAL

 BRICKS

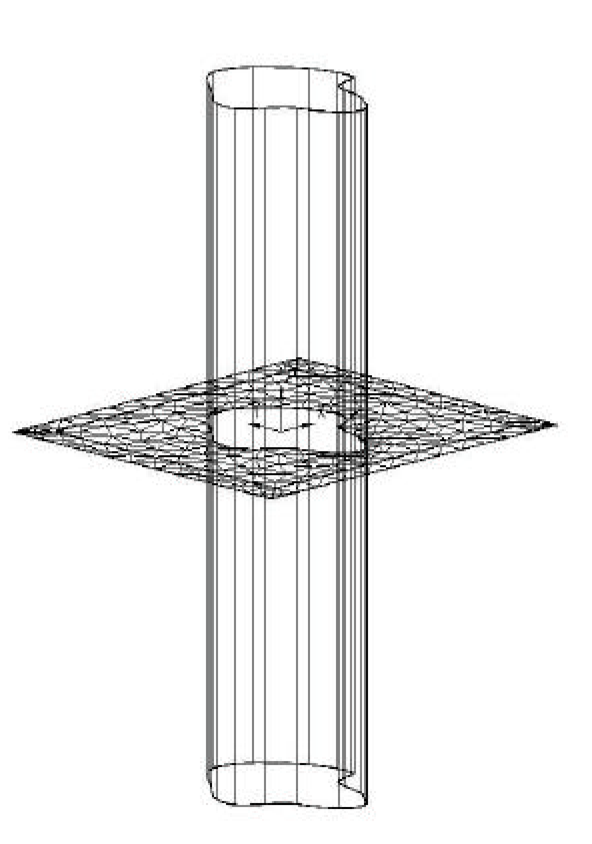
 PRISMS

 TETRAHEDRAL

BOX REPRESENTED BY
RECTANGULAR BRICKS

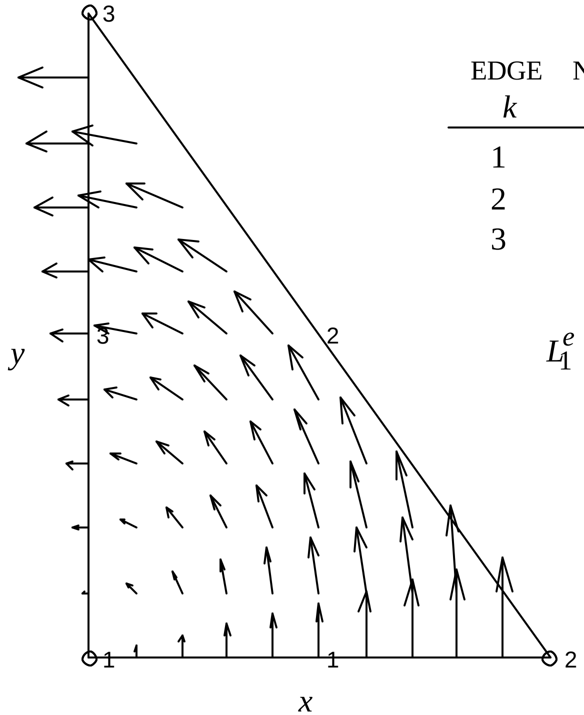


TWO-DIMENSIONAL EXAMPLE:
INFINITE CYLINDER WITH
SURROUNDING MESH



FEM Subdomains (2)

For two-dimensional problems triangular subdomains are used. For triangle *e*, edge *k*:



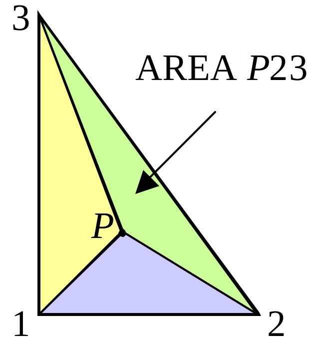
EDGE k	NODE 1 i	NODE 2 j
1	1	2
2	2	3
3	3	1

$$\vec{W}_k^e = \ell_k (L_i^e \nabla L_j^e - L_j^e \nabla L_i^e)$$

$$L_1^e = \frac{\text{AREA } P23}{\text{AREA } 123} \quad L_2^e = \frac{\text{AREA } P31}{\text{AREA } 123} \quad L_3^e = \frac{\text{AREA } P12}{\text{AREA } 123}$$

$P(x,y)$ IS AN INTERNAL POINT

$\ell_k^e =$ LENGTH OF EDGE k

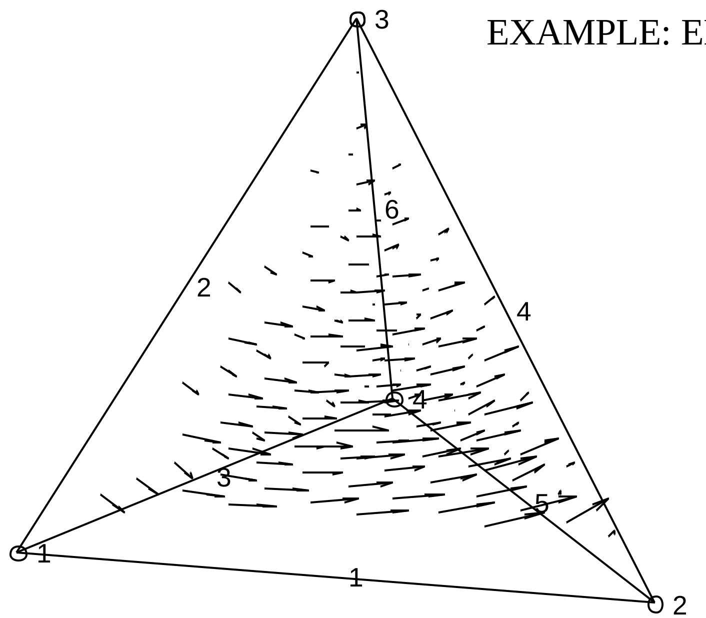


The edge function for edge *n* has only a tangential component across edge *n* and only normal components across the other two edges. The field within triangle *e* is a superposition of the three edge components

$$\vec{E}^e = \sum_{k=1}^{3} E_k^e \vec{W}_k^e$$

FEM Subdomains (3)

For three-dimensional problems, tetrahedra are generally used



EXAMPLE: EDGE ELEMENT ASSOCIATED WITH EDGE 1

$$\vec{W}_1 = \ell_1 (L_1 \nabla L_2 - L_2 \nabla L_1)$$

$$L_2 = \frac{\text{VOLUME P234}}{\text{VOLUME 1234}}$$

$$L_1 = \frac{\text{VOLUME P341}}{\text{VOLUME 1234}}$$

$\ell_1 =$ LENGTH OF EDGE 1

$P =$ POINT AT (x, y, z)

The field associated with edge 6 is shown in the figure. Note that:

1. The field turns around edge 6 (which has endpoints 3 and 4).
2. The field is normal to the planes containing nodes 3 and 4.
3. There is tangential continuity across faces
4. There are six edge elements per tetrahedron. Some may be shared with adjacent tetrahedra.

Plate RCS Using HFSS (1)

The High Frequency Structures Simulator (HFSS) is used to solve transmission line, antenna, and electromagnetic scattering problems using FEM. It has a powerful graphical user's interface (GUI) for building structures, assigning excitations, meshing, computational parameters, and post processing.

Setup for a square plate:

- frequency: 7.5 MHz (40 m wavelength)
- plate dimensions: 200 m by 200 m in the *x-y* plane (5 wavelengths square)
- plate thickness: 0.1 m
- radiation box dimensions: 220 m by 220 m by 40 m

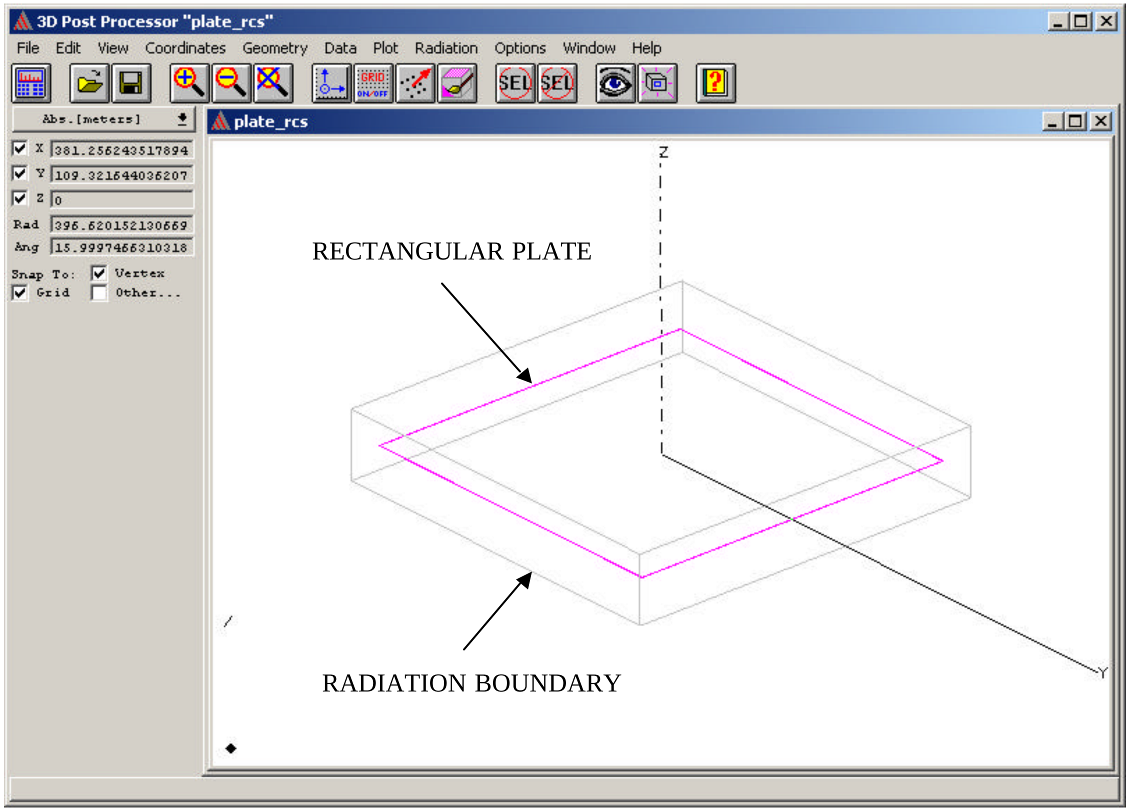
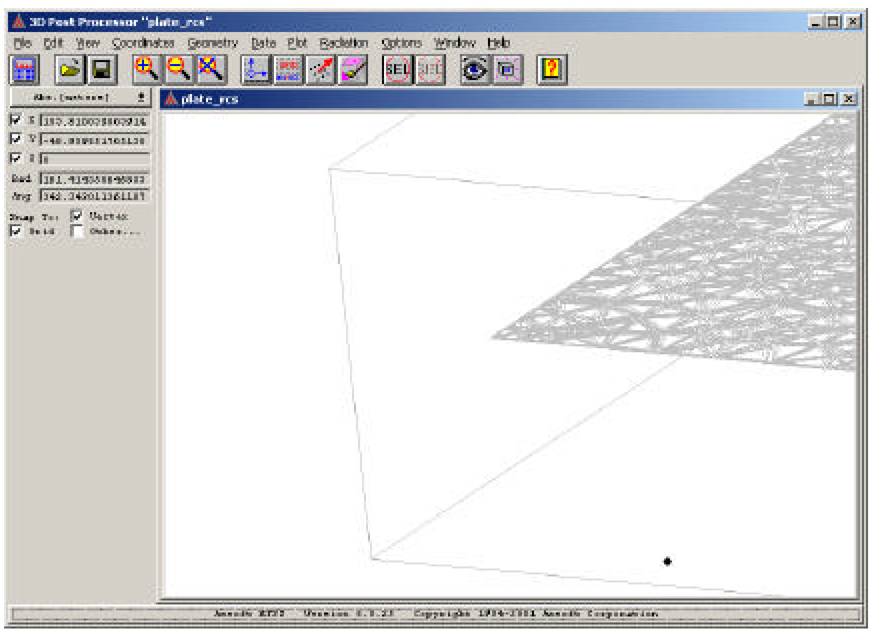


Plate RCS Using HFSS (2)

HFSS has its own computer-aided design (CAD) interface to define the geometry. Drawing data can also be imported from other CAD software packages. HFSS automatically meshes the target and surrounding computational space (the “radiation box”). Radiation boundary conditions are applied on the surfaces of the radiation box.

Close-up of plate mesh



Close-up of radiation box mesh

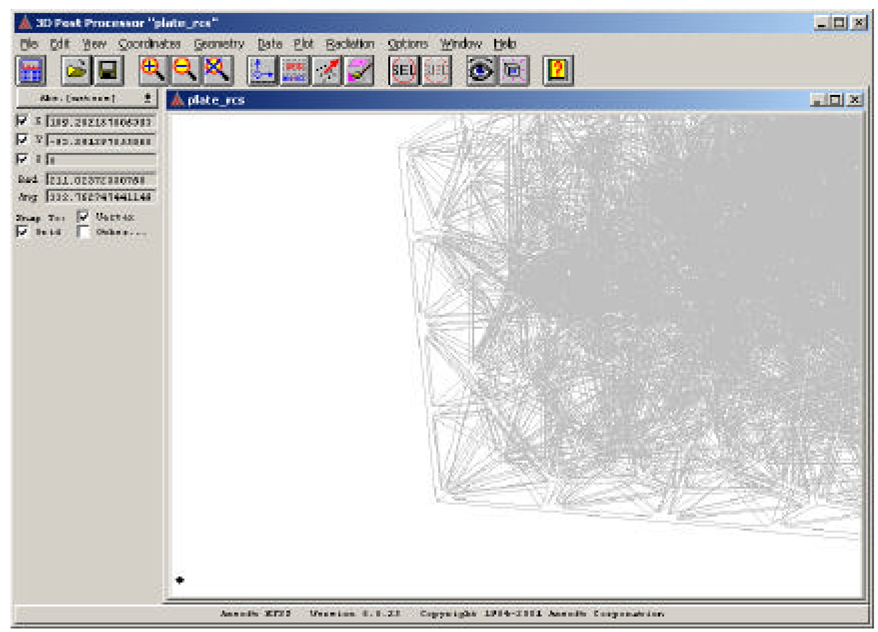


Plate RCS Using HFSS (3)

Bistatic RCS of the square plate for TM polarized $q = 0^\circ$ incidence.

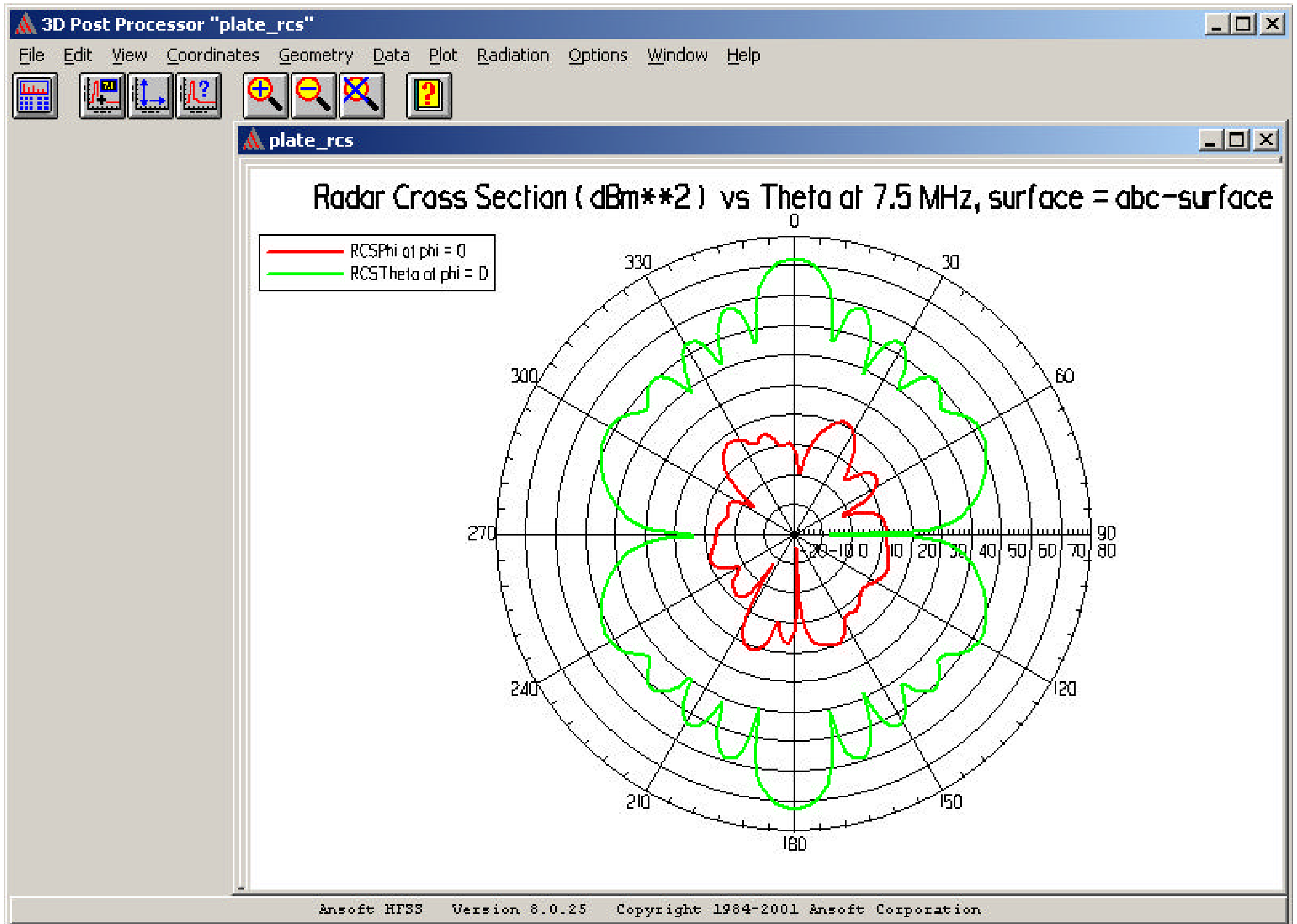
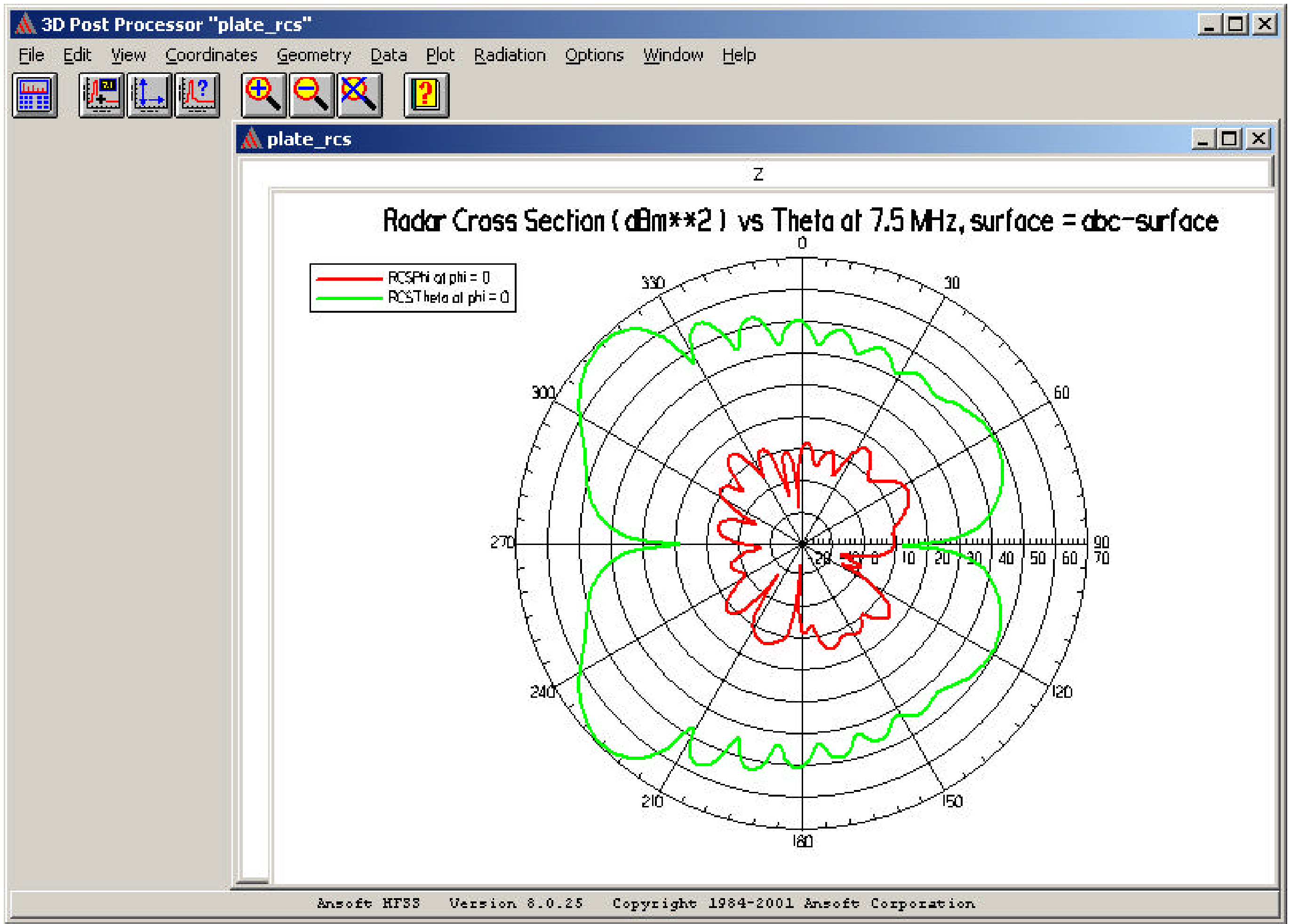


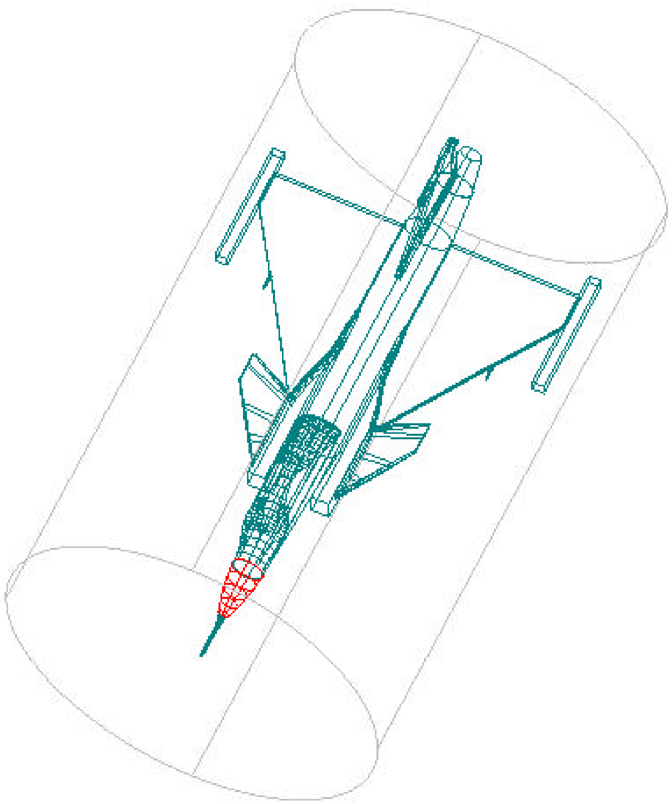
Plate RCS Using HFSS (4)

Bistatic RCS of the square plate for TM polarized $q = 45^\circ$ incidence.

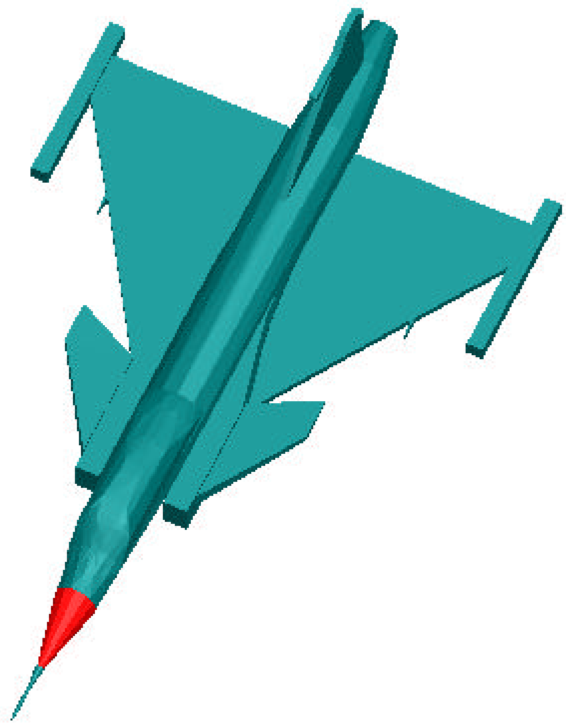


Aircraft Model in HFSS

Gripen aircraft model with cylindrical computational boundary

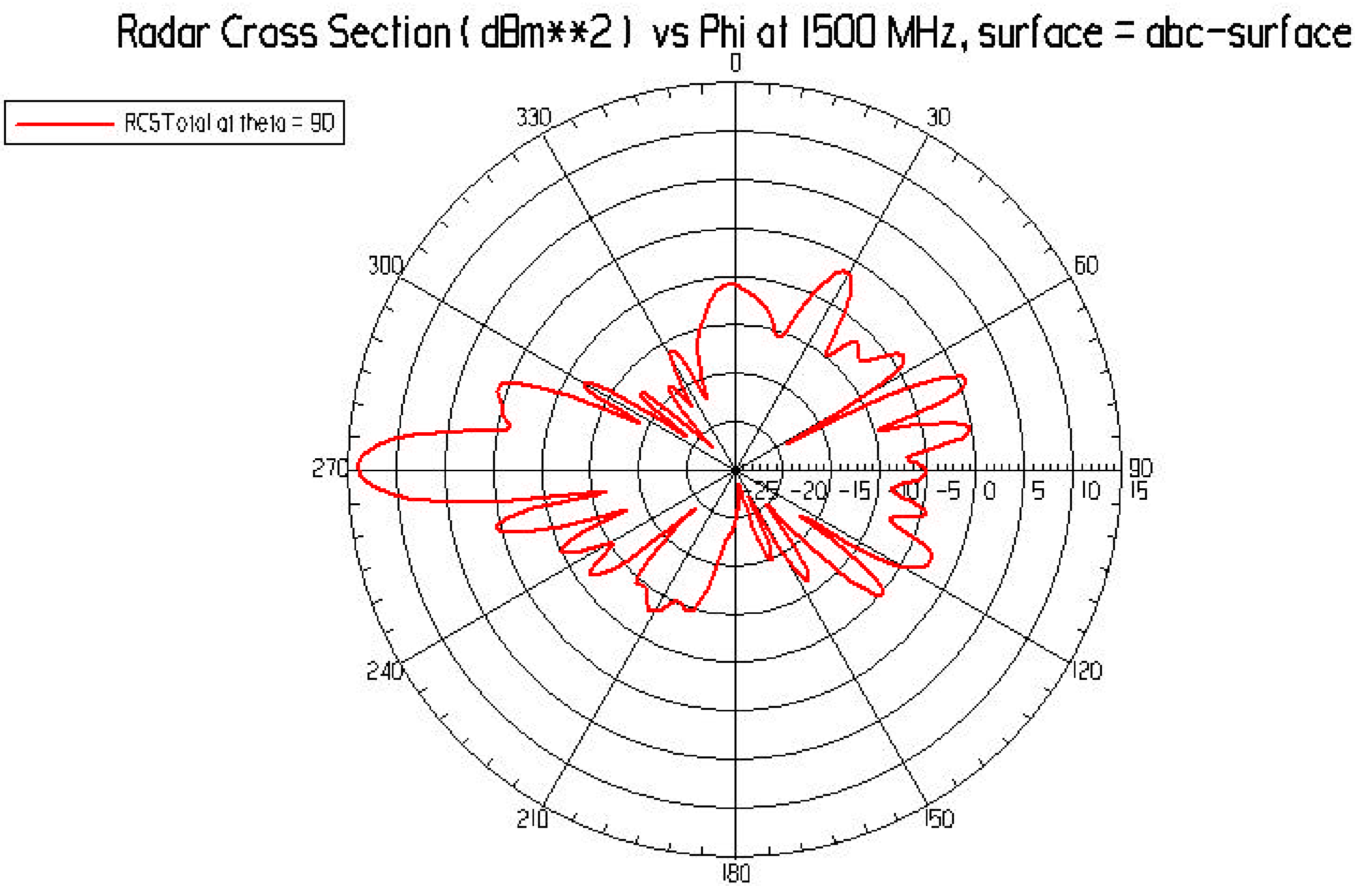


Rendered Aircraft (red is dielectric)



Aircraft Bistatic RCS From HFSS

$$s_{ff} \text{ for } q = f = 90^\circ$$



FEM Summary

- FEM is a frequency domain method
- Advantages include geometric and material adaptability, sparse matrices, and compatibility with other engineering analyses
- Approximate grid termination techniques include
    1. Perfectly matched layers – not physically realizable, but good for computational purposes
    2. Absorbing boundary conditions – complicated when the ABC is applied close to the target surface; simple when far away, but not “node efficient”
- Rigorous termination of the grid requires solving an integral equation for the surface currents on the grid boundary using MM (this is referred to as the finite element – boundary integral method, FE-BI)
    1. Requires solution of the FEM and MM matrix equations simultaneously
    2. The MM partition is dense; the FEM partition is sparse. Therefore the computational intensity has increased substantially.